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Nonlinear Fuzzy Stability of a Functional Equation Related to a Characterization of Inner Product Spaces via Fixed Point Technique

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Abstract. Using the fixed point method, we prove some results concerning the stability of the functional equation

$$\sum_{i=1}^{2n} f(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j) = \sum_{i=1}^{2n} f(x_i) - 2nf(\frac{1}{2n}\sum_{i=1}^{2n} x_i)$$

where f is defined on a vector space and taking values in a fuzzy Banach space, which is said to be a functional equation related to a characterization of inner product spaces.

1. Introduction

Stability problem of functional equations was first posed by Ulam in [35] which was answered by Hyers in [14] for additive mappings. Hyers' result, using unbounded Cauchy different, was generalized for additive mappings in [1] and for linear mappings in [31]. Since then there have been several new results on stability of various classes of functional equations in the Hyers-Ulam sense or Hyers-Ulam-Rassias sense in normed spaces (see [10, 15, 16, 34] and references cited therein). Stability problems of functional equations on arbitrary groups and on non-abelian groups were treated in [6–9]. In [20–23], various stability results concerning Cauchy, Jensen, quadratic and cubic functional equations were investigated in fuzzy normed spaces. Furthermore some stability results concerning additive, quadratic, Cauchy-Jensen, mixed type cubic and quartic functional equations were investigated (cf. [5, 12, 19, 24, 33]) in the setting of non-Archimedean fuzzy normed spaces, non-Archimedean $\mathcal{L}$-fuzzy normed spaces and generalized fuzzy normed spaces, respectively.

It was shown by Rassias [32] that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if for any finite set of vectors $x_1, \dots, x_n \in X$, and a fixed integer $n \geq 2$

$$\sum_{i=1}^{n} \|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\|^2 = \sum_{i=1}^{n} \|x_i\|^2 - n\|\frac{1}{n}\sum_{i=1}^{n} x_i\|^2. \tag{1}$$

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Employing the above identity, Najati and Rassias [26] obtained the functional equation

$$\sum_{i=1}^{n} f(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j) = \sum_{i=1}^{n} f(x_i) - nf(\frac{1}{n}\sum_{i=1}^{n} x_i). \tag{2}$$

It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (2). In [17, 28], the authors introduced the following functional equation

$$\sum_{i=1}^{2n} f(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j) = \sum_{i=1}^{2n} f(x_i) - 2nf(\frac{1}{2n}\sum_{i=1}^{2n} x_i), \tag{3}$$

which is said to be a functional equation related to a characterization of inner product spaces. They obtained the general solution of equation (3) and proved the Hyers-Ulam-Rassias stability of this equation. For notational simplicity, we will denote the functional equation (3) as

$$D_f(x_1, \ldots, x_{2n}) = 0,$$

where D_f is given by

$$D_f(x_1, \ldots, x_{2n}) := \sum_{i=1}^{2n} f(x_i - \frac{1}{2n}\sum_{j=1}^{2n} x_j) - \sum_{i=1}^{2n} f(x_i) + 2nf(\frac{1}{2n}\sum_{i=1}^{2n} x_i). \tag{4}$$

There are many interesting new results concerning functional equations related to inner product spaces have been obtained by Park et al. [27] as well as for fuzzy stability of functional equations related to inner product spaces [11, 29].

The main purpose of this paper is to establish a fuzzy version of the Hyers-Ulam-Rassias stability for the functional equation (3) in fuzzy Banach spaces by using the fixed point method. This paper is different from the earlier papers [11, 29] in the sense that the bound used in this paper for $D_f(x_1, \ldots, x_{2n})$ is quite different from the bound used in [11] (or see [29]). Specializing the function $\varphi(x_1, \ldots, x_{2n})$ which is as a part of the bound, we obtain several results similar to results known for stability of the equation (3) in Banach spaces.

2. Preliminaries

In this section, some definition and preliminary results are given which will be used in this paper. Following [2, 20, 21], we give the following notion of a fuzzy norm.

Definition 2.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$:*
(N1) $N(x, c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.
In this case (X, N) is called a fuzzy normed vector space.

Example 2.2. (cf. [25]). *Let $(X, \|\cdot\|)$ be a normed vector space and $\alpha, \beta > 0$. Then*

$$N(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|}, & t > 0, x \in X, \\ 0, & t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Example 2.3. (cf. [25]). *Let $(X, \|\cdot\|)$ be a normed vector space and $\beta > \alpha > 0$. Then*

$$N(x,t) = \begin{cases} 0, & t \le \alpha\|x\|, \\ \frac{t}{t+(\beta-\alpha)\|x\|}, & \alpha\|x\| < t \le \beta\|x\|, \\ 1, & t > \beta\|x\| \end{cases}$$

is a fuzzy norm on X.

Definition 2.4. (cf. [2, 20, 21]). *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1 (t > 0)$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote by $N - \lim_{n\to\infty} x_n = x$.*

Definition 2.5. (cf. [2, 20, 21]). *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$, there exists $n_0 \in N$ such that $N(x_m - x_n, \delta) > 1 - \varepsilon$ $(m, n \ge n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 2.6. *Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a generalized metric on E if d satisfies*
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$, $\forall x, y \in E$;
(3) $d(x, z) \le d(x, y) + d(y, z)$, $\forall x, y, z \in E$.

The next result is due to Diaz and Margolis [4].

Lemma 2.7. (cf. [4] or [30]). *Let (E, d) be a complete generalized metric space and $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each fixed element $x \in E$, either*

$$d(J^n x, J^{n+1}x) = \infty \;\; \forall n \ge 0,$$
or
$$d(J^n x, J^{n+1}x) < \infty \;\; \forall n \ge n_0,$$

for some natural number n_0. Moreover, if the second alternative holds then:
(i) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(ii) *y^* is the unique fixed point of J in the set $E' := \{y \in E \mid d(J^{n_0}x, y) < +\infty\}$ and $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$, $\forall x, y \in E'$.*

3. Fuzzy Stability of Functional Equations

Before proceeding to the proof of the main results in this section, we shall need the following two lemmas.

Lemma 3.1. (cf. [28]). *Let V and W be real vector spaces. If an odd mapping $f : V \to W$ satisfies (3), then the mapping $f : V \to W$ is additive, that is, f is a solution of $f(x + y) = f(x) + f(y)$ for all $x, y \in V$.*

Lemma 3.2. (cf. [17]). *Let V and W be real vector spaces. If an even mapping $f : V \to W$ satisfies (3), then the mapping $f : V \to W$ is quadratic, that is, f is a solution of $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in V$.*

In this section, we assume that X is a vector space and (Υ, N) is a fuzzy Banach space. We will establish the following stability results for functional equations (3) in fuzzy Banach spaces by using the fixed point method, which is said to be a functional equation related to inner products space. For given mapping $f : X \to \Upsilon$, let $D_f : X^{2n} \to \Upsilon$ be a mapping as defined in (4) for all $x_1, \ldots, x_{2n} \in X$.

Theorem 3.3. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ such that*

$$N(D_f(x_1, \ldots, x_{2n}), t) \geq \frac{t}{t + \varphi(x_1, \ldots, x_{2n})} \tag{5}$$

for all $x_1, \ldots, x_{2n} \in X$. If there exits a constant $0 < L < 1$ such that

$$\varphi(x_1, \ldots, x_{2n}) \leq \frac{L}{2}\varphi(2x_1, \ldots, 2x_{2n}) \tag{6}$$

for all $x_1, \ldots, x_{2n} \in X$, then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that

$$N(f(x) - f(-x) - A(x), t) \geq \frac{n(1 - L)t}{n(1 - L)t + L\Phi(x)} \tag{7}$$

for all $x \in X$ and $t > 0$, where

$$\Phi(x) = \varphi(\underbrace{2x, \ldots, 2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \varphi(\underbrace{-2x, \ldots, -2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}). \tag{8}$$

Proof. Setting $x_1 = \cdots = x_n = x$ and $x_{n+1} = \cdots = x_{2n} = 0$ in (5), we obtain

$$N(3nf(\frac{x}{2}) + nf(\frac{-x}{2}) - nf(x), t) \geq \frac{t}{t + \varphi(\underbrace{x, \ldots, x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})} \tag{9}$$

for all $x \in X$ and all $t > 0$. Replacing x by $-x$ in (9), we get

$$N(3nf(\frac{-x}{2}) + nf(\frac{x}{2}) - nf(-x), t) \geq \frac{t}{t + \varphi(\underbrace{-x, \ldots, -x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})} \tag{10}$$

for all $x \in X$ and all $t > 0$. Thus

$$\begin{aligned}
&N(2n(f(\frac{x}{2}) - f(\frac{-x}{2})) - n(f(x) - f(-x)), 2t)\\
&\geq \min\{N(3nf(\frac{x}{2}) + nf(\frac{-x}{2}) - nf(x), t), N(3nf(\frac{-x}{2}) + nf(\frac{x}{2}) - nf(-x), t)\}\\
&\geq \frac{t}{t + \varphi(\underbrace{x, \ldots, x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \varphi(\underbrace{-x, \ldots, -x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})}
\end{aligned} \tag{11}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) = f(x) - f(-x)$ and

$$\Phi(x) = \varphi(\underbrace{2x, \ldots, 2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \varphi(\underbrace{-2x, \ldots, -2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})$$

for all $x \in X$, we get

$$N(2ng(\frac{x}{2}) - ng(x), 2t) \geq \frac{t}{t + \Phi(\frac{x}{2})} \tag{12}$$

for all $x \in X$ and all $t > 0$. Therefore

$$N(g(2x) - 2g(x), \frac{2}{n}t) \geq \frac{t}{t + \Phi(x)} \tag{13}$$

for all $x \in X$ and all $t > 0$.

Let E_1 be the set of all functions $q_1 : X \to \Upsilon$. Let us introduce a generalized metric on E_1 as follows:

$$d_1(q_1, h_1) := \inf\left\{\lambda \in [0, \infty] \middle| N(q_1(x) - h_1(x), \lambda t) \geq \frac{t}{t + \Phi(x)}, \forall x \in X, \forall t > 0\right\}.$$

It is easy to prove that (E_1, d_1) is a complete generalized metric space [3, 13, 18].

Now we consider the function $\mathcal{J}_1 : E_1 \to E_1$ defined by

$$\mathcal{J}_1 q_1(x) := 2q_1(\frac{x}{2}), \quad \text{for all } q_1 \in E_1 \text{ and } x \in X. \tag{14}$$

Let $q_1, h_1 \in E_1$ and let $\lambda \in [0, \infty]$ be an arbitrary constant with $d_1(q_1, h_1) \leq \lambda$. From the definition of d_1, we have

$$N(q_1(x) - h_1(x), \lambda t) \geq \frac{t}{t + \Phi(x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(\mathcal{J}_1 q_1(x) - \mathcal{J}_1 h_1(x), L\lambda t) &= N(2q_1(\frac{x}{2}) - 2h_1(\frac{x}{2}), L\lambda t) \\ &= N(q_1(\frac{x}{2}) - h_1(\frac{x}{2}), \frac{L}{2}\lambda t) \\ &\geq \frac{\frac{L}{2}t}{\frac{L}{2}t + \Phi(\frac{x}{2})} \geq \frac{\frac{L}{2}t}{\frac{L}{2}t + \frac{L}{2}\Phi(x)} \\ &= \frac{t}{t + \Phi(x)} \end{aligned} \tag{15}$$

for all $x \in X$ and all $t > 0$. Thus

$$d_1(\mathcal{J}_1 q_1, \mathcal{J}_1 h_1) \leq L d_1(q_1, h_1) \tag{16}$$

for all $q_1, h_1 \in E_1$.

It follows from (13) that

$$\begin{aligned} N(g(x) - 2g(\frac{x}{2}), \frac{2}{n}\frac{L}{2}t) &\geq \frac{\frac{L}{2}t}{\frac{L}{2}t + \Phi(\frac{x}{2})} \geq \frac{\frac{L}{2}t}{\frac{L}{2}t + \frac{L}{2}\Phi(x)} \\ &= \frac{t}{t + \Phi(x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, we have $d_1(g, \mathcal{J}_1 g) \leq \frac{L}{n}$. Therefore according to Lemma 2.7, the sequence $\mathcal{J}_1^k g$ converges to a fixed point A of $\mathcal{J}_1$, that is,

$$A : X \to \Upsilon, \quad N - \lim_{n \to \infty} 2^k g(\frac{x}{2^k}) = A(x)$$

and $A(2x) = 2A(x)$ for all $x \in X$. Also A is the unique fixed point $\mathcal{J}_1$ in the set $E_1^* = \{q_1 \in E_1 : d_1(g, q_1) < \infty\}$ and

$$d_1(g, A) \leq \frac{1}{1 - L} d_1(g, \mathcal{J}_1 g) \leq \frac{L}{n(1 - L)},$$

that is, inequality (7) holds true for all $x \in X$ and all $t > 0$. It follows from (5) that

$$N(2^k Dg(\frac{x_1}{2^k}, \ldots, \frac{x_{2n}}{2^k}), 2^k t) \geq \frac{t}{t + \varphi(\frac{x_1}{2^k}, \ldots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \ldots, -\frac{x_{2n}}{2^k})}$$

for all $x_1, \ldots, x_{2n} \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By (5), we get

$$N(2^k Dg(\frac{x_1}{2^k}, \ldots, \frac{x_{2n}}{2^k}), t) \geq \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}(\varphi(\frac{x_1}{2^k}, \ldots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \ldots, -\frac{x_{2n}}{2^k}))}$$

for all $x_1, \ldots, x_{2n} \in X$, all $t > 0$ and all $k \in \mathbb{N}$.

Since $\lim_{k\to\infty} \frac{\frac{t}{2^k}}{\frac{t}{2^k} + \frac{L^k}{2^k}(\varphi(\frac{x_1}{2^k}, \ldots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \ldots, -\frac{x_{2n}}{2^k}))} = 1$ for all $x_1, \ldots, x_{2n} \in X$ and all $t > 0$, we have

$$N(D_A(x_1, \ldots, x_{2n}), t) = 1$$

for all $x_1, \ldots, x_{2n} \in X$ and all $t > 0$. Since every fuzzy Banach space Υ is a real vector space, by Lemma 3.1, the mapping $A : X \to \Upsilon$ is additive. Finally it remains to prove the uniqueness of A. Let $T : X \to \Upsilon$ is another additive mapping satisfying (3) and (7). Since $d_1(g, T) \leq \frac{L}{n(1-L)}$ and T is additive, we get $T \in E_1^*$ and $\mathcal{J}_1 T(x) = 2T(\frac{x}{2}) = T(x)$ for all $x \in X$, i.e., T is a fixed point of $\mathcal{J}_1$. Since A is the unique fixed point of $\mathcal{J}_1$ in E_1^*, then $T = A$. This completes the proof of the theorem. □

Corollary 3.4. *Let $p > 1$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and*

$$N(D_f(x_1, \ldots, x_{2n}), t) \geq \frac{t}{t + \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)} \tag{17}$$

for all $x_1, \ldots, x_{2n} \in X$ and all $t > 0$. Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that

$$N(f(x) - f(-x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^{p+2} \cdot \theta\|x\|^p} \tag{18}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x_1, \ldots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$$

for all $x_1, \ldots, x_{2n} \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result.

Corollary 3.5. *Let $f : X \to \Upsilon$ be an odd mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (5) and (6). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x), t) \geq \frac{n(1-L)t}{n(1-L)t + L\Phi(x)} \tag{19}$$

for all $x \in X$ and $t > 0$, where $\Phi(x)$ is defined in (8).

Theorem 3.6. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^{2n} \to [0, \infty)$ such that*

$$N(D_f(x_1, \ldots, x_{2n}), t) \geq \frac{t}{t + \phi(x_1, \ldots, x_{2n})} \tag{20}$$

for all $x_1, \ldots, x_{2n} \in X$. If there exits a constant $0 < L < 1$ such that

$$\phi(2x_1, \ldots, 2x_{2n}) \leq 2L\phi(x_1, \ldots, x_{2n}) \tag{21}$$

for all $x_1, \ldots, x_{2n} \in X$, then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that

$$N(f(x) - f(-x) - A(x), t) \geq \frac{n(1-L)t}{n(1-L)t + \Psi(x)} \tag{22}$$

for all $x \in X$ and $t > 0$, where

$$\Psi(x) = \phi(\underbrace{2x, \ldots, 2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \phi(\underbrace{-2x, \ldots, -2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}). \tag{23}$$

Proof. Using the same method as in the proof of Theorem 3.3, we have

$$N(g(2x) - 2g(x), \frac{2}{n}t) \geq \frac{t}{t + \Psi(x)} \tag{24}$$

for all $x \in X$ and all $t > 0$, where

$$\Psi(x) = \phi(\underbrace{2x, \ldots, 2x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}) + \phi(\underbrace{-2x, \ldots, -2x}_{n \text{ times}}, \underbrace{0, \ldots, 0}_{n \text{ times}}).$$

We introduce the definitions for E_1 and d_1 as in the proof of Theorem 3.3 (by replacing Φ by Ψ) such that (E_1, d_1) becomes a complete generalized metric space. Now we consider the function $\mathcal{J}_1 : E_1 \to E_1$ defined by

$$\mathcal{J}_1 q_1(x) := \frac{1}{2} q_1(2x), \quad \text{for all } q_1 \in E_1 \text{ and } x \in X. \tag{25}$$

Let $q_1, h_1 \in E_1$ and let $\lambda \in [0, \infty]$ be an arbitrary constant with $d_1(q_1, h_1) \leq \lambda$. From the definition of d_1, we have

$$N(q_1(x) - h_1(x), \lambda t) \geq \frac{t}{t + \Psi(x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(\mathcal{J}_1 q_1(x) - \mathcal{J}_1 h_1(x), L\lambda t) \quad &= N(\frac{1}{2} q_1(2x) - \frac{1}{2} h_1(2x), L\lambda t) \\ &= N(q_1(2x) - h_1(2x), 2L\lambda t) \\ &\geq \frac{2Lt}{2Lt + \Psi(2x)} \geq \frac{2Lt}{2Lt + 2L\Psi(x)} \\ &= \frac{t}{t + \Psi(x)} \end{aligned} \tag{26}$$

for all $x \in X$ and all $t > 0$. So

$$d_1(\mathcal{J}_1 q_1, \mathcal{J}_1 h_1) \leq L d_1(q_1, h_1) \tag{27}$$

for all $q_1, h_1 \in E_1$.

It follows from (24) that

$$N(g(x) - \frac{1}{2} g(2x), \frac{1}{n} t) \geq \frac{t}{t + \Psi(x)}$$

for all $x \in X$ and all $t > 0$. So, we have $d_1(g, \mathcal{J}_1 g) \leq \frac{1}{n}$. Therefore according to Lemma 2.7, the sequence $\mathcal{J}_1^k g$ converges to a fixed point A of $\mathcal{J}_1$, that is,

$$A : X \to \Upsilon, \quad N - \lim_{n \to \infty} \frac{1}{2^k} g(2^k x) = A(x)$$

and $A(2x) = 2A(x)$ for all $x \in X$. Also A is the unique fixed point $\mathcal{J}_1$ in the set $E_1^* = \{q_1 \in E_1 : d_1(g, q_1) < \infty\}$ and

$$d_1(g, A) \leq \frac{1}{1 - L} d_1(g, \mathcal{J}_1 g) \leq \frac{1}{n(1 - L)},$$

that is, inequality (22) holds true for all $x \in X$ and all $t > 0$. The rest of the proof is similar to the proof of Theorem 3.3 and we omit the details. This completes the proof of the theorem. □

Corollary 3.7. *Let $0 < p < 1$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and (17). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that*

$$N(f(x) - f(-x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^{p+2}\cdot\theta\|x\|^p} \tag{28}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.6 by taking

$$\phi(x_1, \ldots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$$

for all $x_1, \ldots, x_{2n} \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result.

Corollary 3.8. *Let $f : X \to \Upsilon$ be an odd mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (20) and (21). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x), t) \geq \frac{n(1-L)t}{n(1-L)t + \Psi(x)} \tag{29}$$

for all $x \in X$ and $t > 0$, where $\Psi(x)$ is defined in (23).

Theorem 3.9. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ such that*

$$N(D_f(x_1, \ldots, x_{2n}), t) \geq \frac{t}{t + \varphi(x_1, \ldots, x_{2n})} \tag{30}$$

for all $x_1, \ldots, x_{2n} \in X$. If there exits a constant $0 < L < 1$ such that

$$\varphi(x_1, \ldots, x_{2n}) \leq \frac{L}{4}\varphi(2x_1, \ldots, 2x_{2n}) \tag{31}$$

for all $x_1, \ldots, x_{2n} \in X$, then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that

$$N(f(x) + f(-x) - Q(x), t) \geq \frac{n(2-2L)t}{n(2-2L)t + L\Phi(x)} \tag{32}$$

for all $x \in X$ and $t > 0$, where $\Phi(x)$ is defined in (8).

Proof. Setting $x_1 = \cdots = x_n = x$ and $x_{n+1} = \cdots = x_{2n} = 0$ in (30), we obtain

$$N(3nf(\frac{x}{2}) + nf(\frac{-x}{2}) - nf(x), t) \geq \frac{t}{t + \varphi(\underbrace{x, \ldots, x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})} \tag{33}$$

for all $x \in X$ and all $t > 0$. Replacing x by $-x$ in (33), we get

$$N(3nf(\frac{-x}{2}) + nf(\frac{x}{2}) - nf(-x), t) \geq \frac{t}{t + \varphi(\underbrace{-x, \ldots, -x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})} \tag{34}$$

for all $x \in X$ and all $t > 0$. Thus

$$\begin{aligned}
&N(4n(f(\frac{x}{2}) + f(\frac{-x}{2})) - n(f(x) + f(-x)), 2t) \\
&\geq \min\{N(3nf(\frac{x}{2}) + nf(\frac{-x}{2}) - nf(x), t), N(3nf(\frac{-x}{2}) + nf(\frac{x}{2}) - nf(-x), t)\} \\
&\geq \frac{t}{t + \varphi(\underbrace{x, \ldots, x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \varphi(\underbrace{-x, \ldots, -x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times})}
\end{aligned} \tag{35}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) = f(x) + f(-x)$ and

$$\Phi(x) = \varphi(\underbrace{2x, \dots, 2x}_{n\ times}, \underbrace{0, \dots, 0}_{n\ times}) + \varphi(\underbrace{-2x, \dots, -2x}_{n\ times}, \underbrace{0, \dots, 0}_{n\ times})$$

for all $x \in X$, we get

$$N(4ng(\frac{x}{2}) - ng(x), 2t) \geq \frac{t}{t + \Phi(\frac{x}{2})} \tag{36}$$

for all $x \in X$ and all $t > 0$. So

$$N(g(2x) - 4g(x), \frac{2}{n}t) \geq \frac{t}{t + \Phi(x)} \tag{37}$$

for all $x \in X$ and all $t > 0$.

Let E_2 be the set of all functions $q_2 : X \to \Upsilon$ and introduce a generalized metric on E_2 as follows:

$$d_2(q_2, h_2) := \inf \left\{ \mu \in [0, \infty] \middle| N(q_2(x) - h_2(x), \mu t) \geq \frac{t}{t + \Phi(x)}, \forall x \in X, \forall t > 0 \right\}.$$

So (E_2, d_2) is a complete generalized metric space. Let $\mathcal{J}_2 : E_2 \to E_2$ defined by

$$\mathcal{J}_2 q_2(x) := 4q_2(\frac{x}{2}), \quad \text{for all } q_2 \in E_2 \text{ and } x \in X. \tag{38}$$

Let $q_2, h_2 \in E_2$ and let $\mu \in [0, \infty]$ be an arbitrary constant with $d(q_2, h_2) \leq \mu$. From the definition of d_2, we have

$$N(q_2(x) - h_2(x), \mu t) \geq \frac{t}{t + \Phi(x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(\mathcal{J}_2 q_2(x) - \mathcal{J}_2 h_2(x), L\mu t) &= N(4q_2(\frac{x}{2}) - 4h_2(\frac{x}{2}), L\mu t) \\ &= N(q_2(\frac{x}{2}) - h_2(\frac{x}{2}), \frac{L}{4}\mu t) \\ &\geq \frac{\frac{L}{4}t}{\frac{L}{4}t + \Phi(\frac{x}{2})} \geq \frac{\frac{L}{4}t}{\frac{L}{4}t + \frac{L}{4}\Phi(x)} \\ &= \frac{t}{t + \Phi(x)} \end{aligned} \tag{39}$$

for all $x \in X$ and all $t > 0$. Therefore

$$d_2(\mathcal{J}_2 q_2, \mathcal{J}_2 h_2) \leq L d_2(q_2, h_2) \tag{40}$$

for all $q, h \in E$.

It follows from (37) that

$$\begin{aligned} N(g(x) - 4g(\frac{x}{2}), \frac{2}{n}\frac{L}{2}t) &\geq \frac{\frac{L}{4}t}{\frac{L}{4}t + \Phi(\frac{x}{2})} \geq \frac{\frac{L}{4}t}{\frac{L}{4}t + \frac{L}{4}\Phi(x)} \\ &= \frac{t}{t + \Phi(x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So, we have $d_2(g, \mathcal{J}_2 g) \leq \frac{L}{2n}$. Therefore according to Lemma 2.7, the sequence $\mathcal{J}_2^k g$ converges to a fixed point Q of $\mathcal{J}_2$, that is,

$$Q : X \to \Upsilon, \quad N - \lim_{n\to\infty} 4^k g(\frac{x}{2^k}) = Q(x)$$

and $Q(2x) = 4Q(x)$ for all $x \in X$. Also Q is the unique fixed point $\mathcal{J}_2$ in the set $E_2^* = \{q_2 \in E_2 : d_2(g, q_2) < \infty\}$ and

$$d_2(g, Q) \leq \frac{1}{1-L} d_2(g, \mathcal{J}_2 g) \leq \frac{L}{2n(1-L)},$$

that is, inequality (32) holds true for all $x \in X$ and all $t > 0$. It follows from (30) that

$$N(4^k Dg(\frac{x_1}{2^k}, \dots, \frac{x_{2n}}{2^k}), 4^k t) \geq \frac{t}{t + \varphi(\frac{x_1}{2^k}, \dots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \dots, -\frac{x_{2n}}{2^k})}$$

for all $x_1, \dots, x_{2n} \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By (30), we get

$$N(4^k Dg(\frac{x_1}{2^k}, \dots, \frac{x_{2n}}{2^k}), t) \geq \frac{\frac{t}{4^k}}{\frac{t}{4^k} + \frac{L^k}{4^k}(\varphi(\frac{x_1}{2^k}, \dots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \dots, -\frac{x_{2n}}{2^k}))}$$

for all $x_1, \dots, x_{2n} \in X$, all $t > 0$ and all $k \in \mathbb{N}$.

Since $\lim_{k\to\infty} \frac{\frac{t}{4^k}}{\frac{t}{4^k} + \frac{L^k}{4^k}(\varphi(\frac{x_1}{2^k}, \dots, \frac{x_{2n}}{2^k}) + \varphi(-\frac{x_1}{2^k}, \dots, -\frac{x_{2n}}{2^k}))} = 1$ for all $x_1, \dots, x_{2n} \in X$ and all $t > 0$, we have

$$N(D_Q(x_1, \dots, x_{2n}), t) = 1$$

for all $x_1, \dots, x_{2n} \in X$ and all $t > 0$. Since every fuzzy Banach space Υ is a real vector space, by Lemma 3.2, the mapping $Q : X \to \Upsilon$ is quadratic. Finally it remains to prove the uniqueness of Q. Let $Q' : X \to \Upsilon$ is another quadratic mapping satisfying (3) and (32). Since $d_2(g, Q') \leq \frac{L}{n(1-L)}$ and Q' is quadratic, we get $Q' \in E_2^*$ and $\mathcal{J}_2 Q'(x) = 4Q'(\frac{x}{2}) = Q'(x)$ for all $x \in X$, that is, Q' is a fixed point of $\mathcal{J}_2$. Since Q is the unique fixed point of $\mathcal{J}_2$ in E_2^*, then $Q' = Q$. This completes the proof of the theorem. □

Corollary 3.10. *Let $p > 2$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and (17). Then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(f(x) + f(-x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^{2+p} \cdot \theta \|x\|^p} \tag{41}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.6 by taking

$$\varphi(x_1, \dots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$$

for all $x_1, \dots, x_{2n} \in X$ and choosing $L = 2^{2-p}$.

Corollary 3.11. *Let $f : X \to \Upsilon$ be an even mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying $f(0) = 0$, (30) and (31). Then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - Q(x), t) \geq \frac{n(2 - 2L)t}{n(2 - 2L)t + L\Phi(x)} \tag{42}$$

for all $x \in X$ and $t > 0$, where $\Phi(x)$ is defined in (8).

Theorem 3.12. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\phi : X^{2n} \to [0, \infty)$ such that*

$$N(D_f(x_1, \ldots, x_{2n}), t) \geq \frac{t}{t + \phi(x_1, \ldots, x_{2n})} \tag{43}$$

for all $x_1, \ldots, x_{2n} \in X$. If there exits a constant $0 < L < 1$ such that

$$\phi(2x_1, \ldots, 2x_{2n}) \leq 4L\phi(x_1, \ldots, x_{2n}) \tag{44}$$

for all $x_1, \ldots, x_{2n} \in X$, then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that

$$N(f(x) + f(-x) - Q(x), t) \geq \frac{n(2 - 2L)t}{n(2 - 2L)t + \Psi(x)} \tag{45}$$

for all $x \in X$ and $t > 0$, where where $\Psi(x)$ is defined in (23).

Proof. Using the same method as in the proof of Theorem 3.9, we have

$$N(g(2x) - 4g(x), \frac{2}{n}t) \geq \frac{t}{t + \Psi(x)} \tag{46}$$

for all $x \in X$ and all $t > 0$, where

$$\Psi(x) = \phi(\underbrace{2x, \ldots, 2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}) + \phi(\underbrace{-2x, \ldots, -2x}_{n\ times}, \underbrace{0, \ldots, 0}_{n\ times}).$$

We introduce the definitions for E_2 and d_2 as in the proof of Theorem 3.9 (by replacing Φ by Ψ) such that (E_2, d_2) becomes a complete generalized metric space. Now we consider the function $\mathcal{J}_2 : E_2 \to E_2$ defined by

$$\mathcal{J}_2 q_2(x) := \frac{1}{4} q_2(2x), \quad \text{for all } q_2 \in E_2 \text{ and } x \in X. \tag{47}$$

Let $q_2, h_2 \in E_2$ and let $\mu \in [0, \infty]$ be an arbitrary constant with $d_2(q_2, h_2) \leq \mu$. From the definition of d_2, we have

$$N(q_2(x) - h_2(x), \mu t) \geq \frac{t}{t + \Psi(x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(\mathcal{J}_2 q_2(x) - \mathcal{J}_2 h_2(x), L\mu t) &= N(\frac{1}{4} q_2(2x) - \frac{1}{4} h_2(2x), L\mu t) \\ &= N(q_2(2x) - h_2(2x), 4L\mu t) \\ &\geq \frac{4Lt}{4Lt + \Psi(2x)} \geq \frac{4Lt}{4Lt + 4L\Psi(x)} \\ &= \frac{t}{t + \Psi(x)} \end{aligned} \tag{48}$$

for all $x \in X$ and all $t > 0$. Therefore

$$d_2(\mathcal{J}_2 q_2, \mathcal{J}_2 h_2) \leq L d_2(q_2, h_2) \tag{49}$$

for all $q_2, h_2 \in E_2$.

It follows from (46) that

$$N(g(x) - \frac{1}{4} g(2x), \frac{1}{2n} t) \geq \frac{t}{t + \Psi(x)}$$

for all $x \in X$ and all $t > 0$. So, we have $d_2(g, \mathcal{J}_2 g) \leq \frac{1}{2n}$. Therefore according to Lemma 2.7, the sequence $\mathcal{J}_2^k g$ converges to a fixed point Q of $\mathcal{J}_2$, that is,

$$Q : X \to \Upsilon, \quad N - \lim_{n\to\infty} \frac{1}{4^k} g(2^k x) = Q(x)$$

and $Q(2x) = 4Q(x)$ for all $x \in X$. Also Q is the unique fixed point $\mathcal{J}_2$ in the set $E_2^* = \{q_2 \in E_2 : d_2(g, q_2) < \infty\}$ and

$$d_2(g, Q) \leq \frac{1}{1-L} d_2(g, \mathcal{J}_2 g) \leq \frac{1}{2n(1-L)},$$

that is, inequality (45) holds true for all $x \in X$ and all $t > 0$. The rest of the proof is similar to the proof of Theorem 3.9 and we omit the details. This completes the proof of the theorem. □

Corollary 3.13. *Let $0 < p < 2$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and (17). Then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(f(x) + f(-x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^{2+p} \cdot \theta\|x\|^p} \tag{50}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.9 by taking

$$\phi(x_1, \ldots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$$

for all $x_1, \ldots, x_{2n} \in X$. Next choosing $L = 2^{p-2}$, we get the desired result. □

Corollary 3.14. *Let $f : X \to \Upsilon$ be an even mapping for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying $f(0) = 0$, (43) and (44). Then there exists a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - Q(x), t) \geq \frac{n(2-2L)t}{n(2-2L)t + \Psi(x)} \tag{51}$$

for all $x \in X$ and $t > 0$, where $\Phi(x)$ is defined in (23).

Combining Theorems 3.3 and 3.9, we obtain the following result.

Theorem 3.15. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (5) and (31). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) and a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x) - Q(x), t) \geq \frac{n(1-L)t}{n(1-L)t + 3L\Phi(x)} \tag{52}$$

for all $x \in X$ and $t > 0$, where $\Phi(x)$ is defined in (8).

Proof. By Theorems 3.3 and 3.9, we obtain

$$N(f(x) - f(-x) - A(x), t) \geq \frac{n(1-L)t}{n(1-L)t + L\Phi(x)},$$

$$N(f(x) + f(-x) - Q(x), t) \geq \frac{n(2-2L)t}{n(2-2L)t + L\Phi(x)}$$

for all $x \in X$ and $t > 0$. Thus

$$\begin{aligned} &N(2f(x) - A(x) - Q(x), 2t) \\ &\geq \min\{N(f(x) - f(-x) - A(x), t), N(f(x) + f(-x) - Q(x), t\} \\ &\geq \frac{n(2-2L)t}{n(2-2L)t + 3L\Phi(x)} \end{aligned}$$

for all $x \in X$ and $t > 0$, and we get the desired result. □

Corollary 3.16. *Let $p > 2$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and (17). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) and a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 3 \cdot 2^{3+p} \cdot \theta\|x\|^p} \tag{53}$$

for all $x \in X$ and all $t > 0$.

Proof. Define $\varphi(x_1, \ldots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$, $L = 2^{2-p}$, and apply Theorem 3.15 to get the desired result. □

Similarly, combining Theorems 3.6 and 3.12, we obtain the following result.

Theorem 3.17. *Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{2n} \to [0, \infty)$ satisfying (20) and (21). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) and a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x) - Q(x), t) \geq \frac{n(1 - L)t}{n(1 - L)t + 3\Psi(x)} \tag{54}$$

for all $x \in X$ and $t > 0$, where $\Psi(x)$ is defined in (23).

Proof. Similar to the proof of Theorem 3.15, the result follows from Theorems 3.6 and 3.12. □

Corollary 3.18. *Let $0 < p < 1$ and θ be non-negative real numbers. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to \Upsilon$ be a mapping satisfying $f(0) = 0$ and (17). Then there exists a unique additive mapping $A : X \to \Upsilon$ satisfying (3) and a unique quadratic mapping $Q : X \to \Upsilon$ satisfying (3) such that*

$$N(2f(x) - A(x) - Q(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 3 \cdot 2^{2+p} \cdot \theta\|x\|^p} \tag{55}$$

for all $x \in X$ and all $t > 0$.

Proof. Define $\varphi(x_1, \ldots, x_{2n}) = \theta(\|x_1\|^p + \cdots + \|x_{2n}\|^p)$, $L = 2^{p-1}$, and apply Theorem 3.17 to get the desired result. □

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